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Veiled assonance between geodesics on ellipsoid and billiard in its focal ellipse

M.B. Tabanov [1]

Chair of Applied Mathematics, St.-Petersburg State Academy of Aerospace Instrumentations, Bolshaja Morskaja 67, St.-Petersburg 190000, Russia

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Abstract

We introduce new special ellipsoidal confocal coordinates in $\mathbb{R}^n$ ($n \geq 3$) and apply them to the geodesic problem on a triaxial ellipsoid in $\mathbb{R}^3$ as well as to the billiard problem in its focal ellipse.

Using such appropriate coordinates we show that these different dynamical systems have the same common analytic first integral. This fact is not evident because there exists a geometrical spatial gap between the geodesic and billiard flows under consideration, and this separating gap just "veils" the resemblance of the two systems.

In short, a geodesic on the ellipsoid and a billiard trajectory inside its focal ellipse are in a "veiled assonance"—under the same initial data they will be tangent to the same confocal hyperboloid. But this assonance is rather incomplete: the dynamical systems in question differ by their intrinsic action angle-variables, thereby the different dynamics arise on the same phase space (i.e. the same phase curves in the same phase space bear quite different rotation numbers).

Some results of this work have been published before in Russian (Tabanov, 1993) and presented to the International Geometrical Colloquium (Moscow, May 10–14, 1993) and the International Symposium on Classical and Quantum Billiards (Ascona, Switzerland, July 25–30, 1994).

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1. Introduction

A dynamical problem of free motion of the point mass on the surface of the ellipsoid in the finite-dimensional Euclidean space is well known from 1839, when using the invented

[1] E-mail: tabanov@misha.usr.saai.ru

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ellipsoidal coordinates, C. Jacobi succeeded in separating the variables in the equation of geodesics (Jacobi, 1839, 1841, 1866).

On the other hand, Birkhoff first formulated in 1927, a billiard problem for the planar domain, bounded by the smooth strictly convex closed curve (Birkhoff, 1927a, b). The point follows a straight-line path with the unit velocity inside the domain and reflects from the boundary $\partial\Omega$ according to the law "the angle of incidence is equal to angle of reflection".

To describe a corresponding billiard mapping Birkhoff suggested the use of the variables (s, u), where $s = s(\boldsymbol{x}) \mathrm{mod}(|\partial\Omega|)$ denotes an oriented arclength of $\partial\Omega$ between some fixed point $\boldsymbol{x}_c \in \partial\Omega$ and a point $\boldsymbol{x} \in \partial\Omega$ of reflection on the boundary; $|\partial\Omega|$ is a length of $\partial\Omega$, and $u = -\cos\theta$, where θ is the angle between a positively oriented tangent vector to $\partial\Omega$ at the point $\boldsymbol{x}$ and a given vector of velocity in this point, $0 \le \theta \le \pi$. A mapping $T : (s_0, u_0) \to (s_1, u_1)$ is the diffeomorphism of a corresponding phase space $S^1 \times [-1, 1]$ and preserves a measure $\mathrm{d}m = \mathrm{d}s \cdot \mathrm{d}u$ (Birkhoff, 1927a, b; Lazutkin, 1981; Sinay, 1977).

For the special case when $\partial\Omega$ is an ellipse, the billiard mapping T admits an analytic first integral and Birkhoff described the qualitative behaviour of T in this case (see also Abdrahmanov (1990); Amiran (1988); Bolotin (1990); Keller and Rubinow (1960); Moser (1980); Ramani et al. (1986); Stepin (1981) Tabachnikov (1995); Veselov (1991)).

We consider the problems of both Jacobi and Birkhoff from a common point of view. We use such elliptic coordinates on the plane which allow us to obtain explicit formulas for the billiard mapping in the ellipse and effectively investigate its properties. A natural generalization of these elliptic coordinates in the case of higher dimensions gives rise to new special ellipsoidal coordinates in $\mathbb{R}^n$, $n \ge 3$, which make it possible by the way, to describe with convenient formulas a geodesic flow on the surface of the ellipsoid.

2. Elliptic billiard

Firstly, let us consider Birkhoff's billiard inside an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad 0 < b^2 < a^2. \tag{2.1}$$

We shall use in $\mathbb{R}^2$ the well-known elliptic coordinates (μ, φ) :

$$x = h \cdot \cosh\mu \cdot \cos\varphi, \qquad y = h \cdot \sinh\mu \cdot \sin\varphi, \tag{2.2}$$

where $h^2 = a^2 - b^2$, $0 \le \mu < \infty$, $0 \le \varphi < 2\pi$. The two families of confocal ellipses $\mu = \text{const}$ and hyperbolas $\varphi = \text{const}$ form an orthogonal net of the curves. The ellipse $\partial\Omega$ under consideration has the equation $\mu = \mu_0$, where $\cosh^2\mu_0 = a^2/h^2$, its eccentricity is $\epsilon_0 = \cosh^{-1}\mu_0$ and an arclength $\mathrm{d}s = h\sqrt{\cosh^2\mu_0 - \cos^2\varphi} \cdot \mathrm{d}\varphi$. Let us identify the points of the ellipse $\partial\Omega$, which are symmetric with respect to the origin $(x, y) = (0, 0)$ and so we put $0 \le \varphi < \pi$. We obtain explicit formulas for the billiard mapping $T : (\varphi_0, \theta_0) \mapsto (\varphi_1, \theta_1)$ in the ellipse $\mu = \mu_0$, using Birkhoff's method (Birkhoff, 1927a):

$$T : (\varphi_0, \theta_0) \mapsto (\varphi_1, \theta_1),$$

$$\varphi_1 = -\varphi_0 + 2 \arctan\left(\tanh \mu_0 \cdot \frac{\tan \varphi_0 + \tan \theta_0 \cdot \tanh \mu_0}{\tanh \mu_0 - \tan \varphi_0 \cdot \tan \theta_0}\right) (\mathrm{mod}\, \pi), \tag{2.3}$$

$$\theta_1 = -\theta_0 + \arctan\left(\frac{\tanh \mu_0}{\tan \varphi_0}\right) - \arctan\left(\frac{\tanh \mu_0}{\tan \varphi_1}\right) (\mathrm{mod}\, \pi).$$

The mapping (2.3) is reversible under the involution $R : (\varphi, \theta) \mapsto (\varphi, \pi - \theta) = (\varphi, -\theta)$, i.e., $T^{-1} = R \circ T \circ R$, $R^2 = \mathrm{Id}$. By virtue of mapping $\hat{P} : (\varphi, \theta) \mapsto (-\varphi, -\theta)(\mathrm{mod}\, \pi)$ commutes with T, i.e., $\hat{P} \circ T = T \circ \hat{P}$, an involution $G = R \circ \hat{P} : (\varphi, \theta) \mapsto (\pi - \varphi, \theta) = (-\varphi, \theta)$ reverses T too. We obtain T as a composition of two involutions: $T = R \circ (R \circ T)$ or $T = G \circ (G \circ T)$.

The fixed points of the mapping T on the boundary $\partial\Omega$ correspond to the values $\theta = 0, \pi$. A point $H = (\varphi, \theta) = (0, \frac{1}{2}\pi)$ is a hyperbolic fixed point of T and it is responsible for the trajectory of the billiard along the major axis of the ellipse $\mu = \mu_0$. A fixed point $E = (\varphi, \theta) = (\frac{1}{2}\pi, \frac{1}{2}\pi)$ belongs to the elliptic type and it corresponds to the trajectory along the minor axis. The eigenvalues of the linear part of the mapping at the hyperbolic fixed point H has the form

$$\lambda_1 = \frac{\cosh \mu_0 + 1}{\cosh \mu_0 - 1} > 1, \qquad \lambda_2 = \lambda_1^{-1}. \tag{2.4}$$

A billiard flow S_t (Sinay, 1977) in the ellipse $\partial\Omega$ is an integrable one and we obtain its first integral I_t in the form

$$I_t = \cosh^2 \mu \cdot \cos^2 \theta + \cos^2 \varphi \cdot \sin^2 \theta, \tag{2.5}$$

where $0 \le \mu \le \mu_0$, $0 \le \varphi < \pi$ and θ is an angle between a given trajectory of the billiard flow and a tangent vector to the ellipse of the confocal family $\mu = \mathrm{const}$ at the intersection point with the trajectory, with the corresponding value of parameter t along the billiard trajectory. Since the first integral I_t of the flow does not depend on the parameter t, we obtain the first integral I of the billiard mapping T in the form (2.5) putting $\mu = \mu_0$, and one can verify that the mapping T also preserves a symplectic 2-form $\mathrm{d}\varphi \wedge \mathrm{d}\theta$ besides the measure $\mathrm{d}m$ due to the integrability.

A value $I = 0$ corresponds to the elliptic fixed point E. If $0 < I < 1$, then the trajectories of the flow S_t cross the part of the major axis of $\partial\Omega$ between its foci, and they (or their continuations) will be tangent to the same confocal hyperbolas $\varphi = \pm\varphi_c(\mathrm{mod}\, \pi)$, $\cos^2 \varphi_c = I$, after each reflection from the boundary. A value $I = 1$ corresponds to the union of two branches of the stable $W_0^{\mathrm{s}}(H)$ and unstable $W_0^{\mathrm{u}}(H)$ manifolds ("separatrices") of the point H,

$$W_0(H) = W_0^{\mathrm{s,u}}(H) = \{(\varphi, \theta) \colon T^n(\varphi, \theta) \to H \text{ as } n \to \pm\infty\}; \tag{2.6}$$

these manifolds coincide with one another and they are responsible for the trajectories through the foci of the ellipse. If $1 < I < \cosh^2 \mu_0$, then the trajectories of the billiard flow after each reflection from $\partial\Omega$ will be tangent to the same confocal ellipse $\mu = \mu_c$, where $\cosh^2 \mu_c = I$. Finally, $I = \cosh^2 \mu_0$ corresponds to the fixed points of T on the boundary $\partial\Omega$ (i.e. $\theta = 0, \pi$).

Let us introduce by analogy with Birkhoff (see, Arnold (1978); Arnold et al. (1988); Birkhoff (1927a, b); Poincaré (1892)) the action angle-variables (κ, I) in terms of which the transformation T_0 takes the form

$$\kappa_1 = \kappa_0 + \alpha(I_0) (\operatorname{mod} \pi), \qquad I_1 = I_0,$$

where α is an analytic function. Here $\alpha(I)$ is called *rotation number* of the invariant curve $I = \text{const}$ (Birkhoff, 1927b). The relation between variables φ and κ has the form

$$\varphi = \arccos\left[\sqrt{I} \cdot sn\left(F\left(\sqrt{I}\right)\left(1 - \frac{2}{\pi}\kappa\right)\right)\right], \quad \text{under } 0 \leq I < 1,$$

and

$$\varphi = \arccos\left[sn\left(F\left(\frac{1}{\sqrt{I}}\right)\left(1 - \frac{2}{\pi}\kappa\right)\right)\right], \quad \text{under } 1 < I < \cosh^2 \mu_0,$$

where sn is one of the Jacobian elliptic functions (Whittaker and Watson, 1927) and

$$F(k) = \int_0^{\pi/2} \frac{\mathrm{d}\tau}{\sqrt{1 - k^2 \sin^2 \tau}}.$$

We calculate the rotation numbers $\alpha(I)$ on the invariant curves $I = \text{const}$ of the mapping T_0 and obtain under $0 \leq I < 1$:

$$\alpha(I) = \frac{\pi}{2F(\sqrt{I})} \cdot F\left(\arcsin\left(\frac{2 \tanh \mu_0 \cdot \sqrt{\cosh^2 \mu_0 - I}}{\cosh^2 \mu_0 - I + I \cdot \tanh^2 \mu_0}\right), \sqrt{I}\right), \tag{2.7}$$

where

$$F(z, k) = \int_0^{z} \frac{\mathrm{d}\tau}{\sqrt{1 - k^2 \sin^2 \tau}}.$$

Note that

$$\alpha(0) = \arcsin\left(\frac{2 \cdot \tanh \mu_0}{\cosh \mu_0}\right).$$

If $1 < I < \cosh^2 \mu_0$, then

$$\alpha(I) = \frac{\pi}{2F(1/\sqrt{I})} \times F\left(\arcsin\left(\sqrt{I} \cdot \frac{2 \tanh \mu_0 \cdot \sqrt{\cosh^2 \mu_0 - I}}{\cosh^2 \mu_0 - I + I \cdot \tanh^2 \mu_0}\right), 1/\sqrt{I}\right). \tag{2.8}$$

"The transformation of each invariant curve (I = const) of the analytic family which abuts on $\theta = 0$ is essentially a rotation of that curve into itself through an angle α which varies analytically with the curve... But this variable is not to be regarded as defined along the limiting non-analytic curve of course" (Birkhoff, 1927b). However, it is possible to consider the prolongation of the function $\alpha(I)$ by continuity onto this limiting value $I = 1$. We find that $\alpha(1) = 0 (\operatorname{mod} \pi)$. Note, that if $I = \cosh^2 \mu_0$, then $\alpha(I) = 0$.

If an invariant curve I is such that $\alpha(I)$ is commensurable with π, then corresponding trajectories of the billiard flow S_t are periodical; they all close up after the same number of reflections from the boundary and have the same length, and so one can prove once more the famous "Poncelet Porism" (Berger, 1990a; Cayley, 1858; Chang and Friedberg, 1988; Griffiths and Harris, 1978; Poncelet, 1866; Tabachnikov, 1995).

It is possible to obtain explicit formulas for

$$T_0^n(\varphi_0, \theta_0) = (\varphi_n(\varphi_0, \theta_0), \theta_n(\varphi_0, \theta_0))$$

under integer $n \geq 1$. We find for $0 \leq I < 1$:

$$\varphi_n = \arccos\left[\sqrt{I} \cdot sn\left(F\left(\arcsin\left(\frac{\cos \varphi_0}{\sqrt{I}}\right), \sqrt{I}\right) - n \cdot \delta_1\right)\right], \tag{2.9}$$

where

$$\delta_1 = F\left(\arcsin\left(\frac{2 \tanh \mu_0 \cdot \sqrt{\cosh^2 \mu_0 - I}}{\cosh^2 \mu_0 - I + I \cdot \tanh^2 \mu_0}\right), \sqrt{I}\right).$$

If $1 < I < \cosh^2 \mu_0$, then

$$\varphi_n = \arccos\left[sn\left(F(\tfrac{1}{2}\pi - \varphi_0, 1/\sqrt{I}) - n \cdot \delta_2\right)\right], \tag{2.10}$$

where

$$\delta_2 = F\left(\arcsin\left(\sqrt{I} \cdot \frac{2 \tanh \mu_0 \cdot \sqrt{\cosh^2 \mu_0 - I}}{\cosh^2 \mu_0 - I + I \cdot \tanh^2 \mu_0}\right), 1/\sqrt{I}\right).$$

In a particular case, under $I = 1$ we find

$$\varphi_n = \pm 2 \arctan(\mathrm{e}^{n \ln \lambda_1} \cdot \tan \tfrac{1}{2}\varphi_0) (\operatorname{mod} \pi) \tag{2.11}$$

on the coinciding separatrices $W_0(H) \equiv W_0^{\mathrm{u,s}}(H)$, having equation

$$\tan^2 \theta = \frac{\sinh^2 \mu_0}{\sin^2 \varphi}. \tag{2.12}$$

We obtain $\theta_n(\varphi_0, \theta_0)$, using formula (2.5) for the first integral I of the mapping T_0.

One can consider the parametric representations of the curves $W_0^{\mathrm{u,s}}(H)$ in the form

$$\begin{aligned} \varphi^{\mathrm{u,s}} &= \varphi_\pm(t) = \pm \arctan[\sinh^{-1}(t \ln \lambda_1)] (\operatorname{mod} \pi), \\ \theta^{\mathrm{u,s}} &= \theta_\pm(t) = \pm \arctan[\sinh \mu_0 \cdot \cosh(t \ln \lambda_1)] (\operatorname{mod} \pi), \end{aligned} \tag{2.13}$$

then $(\varphi_\pm(t), \theta_\pm(t))$ will be the separatrices solution of the system (2.3), satisfying the normalizing condition $\varphi_\pm(0) = \frac{1}{2}\pi (\operatorname{mod} \pi)$.

Let us put in (2.3), $\varphi_k = \varphi(t+k)$, $\theta_k = \theta(t+k)$, $k = 0, 1$ and note that the billiard mapping onto separatrices (2.13) is conjugated with a shift $t \mapsto t+1$. Then we consider for such an obtained system of Eq. (2.3) a corresponding linear system of "equations in variations" along the solution (2.13). We may rewrite the latter system in the form of the second-order linear homogeneous difference equation depending on t coefficients:

$$\begin{aligned} &B(t-1)\cdot\varphi(t+1) + B(t)\cdot\varphi(t-1) \\ &\quad +[B(t) - 2B(t-1)B(t)D(t) + B(t-1)]\cdot\varphi(t) = 0, \end{aligned} \tag{2.14}$$

where

$$B(t) = \frac{2}{\tanh \mu_0}\cdot\left[1 - \frac{1}{\sinh^2 \mu_0 \cdot \cosh(t\ln\lambda_1)\cdot\cosh((t+1)\ln\lambda_1)}\right],$$

$$D(t) = \frac{\cosh \mu_0 \cdot \sinh \mu_0 \cdot \cosh^2(t\ln\lambda_1)}{1 + \sinh^2 \mu_0 \cdot \cosh^2(t\ln\lambda_1)}.$$

The proof of the following lemma is straightforward.

Lemma 1. *General solution of Eq.* (2.14) *has the form*

$$\varphi(t) = a(t)\varphi_1(t) + b(t)\varphi_2(t),$$

where $a(t)$, $b(t)$ *are the arbitrary periodic functions with period* 1,

$$\begin{aligned} \varphi_1(t) &= \frac{1}{\cosh(t\ln\lambda_1)}, \\ \varphi_2(t) &= \sinh(t\ln\lambda_1) + \frac{t\cdot\sinh(\ln\lambda_1)}{\cosh(t\ln\lambda_1)}, \end{aligned} \tag{2.15}$$

and the Kazorati determinant $W(\varphi_1, \varphi_2)(t)$ *of this basic solution is equal to*

$$W(\varphi_1, \varphi_2)(t) = \frac{2}{\sinh \mu_0}\cdot B(t).$$

Recall that the Kazorati determinant $W(\varphi_1, \varphi_2)(t)$ of two basic solutions is defined as

$$W(\varphi_1, \varphi_2)(t) = \varphi_1(t)\Delta\varphi_2(t) - \varphi_2(t)\Delta\varphi_1(t),$$

where $\Delta\varphi(t) = \varphi(t+1) - \varphi(t)$.

It is very interesting to underline the remarkable correspondence between (2.15) and the well-known basic solutions

$$u_1(t) = \frac{1}{\cosh(t)}, \qquad u_2(t) = \sinh(t) + \frac{t}{\cosh(t)}$$

of the linear homogeneous second-order differential equation in variations

$$\frac{d^2}{dt^2}u(t) + \left[\frac{2}{\cosh^2(t)} - 1\right]\cdot u(t) = 0$$

for the non-linear pendulum equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \sin x(t) = 0$$

along its separatrices' solution $x(t) = 2\arctan(\sinh(t))$.

3. New ellipsoidal confocal coordinates in $\mathbb{R}^3$

Let us turn to the geodesic problem on a general triaxial ellipsoid E in $\mathbb{R}^3$, having the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad 0 < c^2 < b^2 < a^2 \tag{3.1}$$

and introduce in $\mathbb{R}^3$ the special ellipsoidal confocal coordinates (ψ, μ, φ) by analogy with (2.2):

$$\begin{aligned} x &= \frac{h}{\cosh \mu_0} \cdot \cosh \psi \cdot \cosh \mu \cdot \cos \varphi, \\ y &= \frac{h}{\sinh \mu_0} \cdot \sinh \psi \cdot \sinh \mu \cdot \sin \varphi, \\ z &= \pm \frac{h}{\cosh \mu_0 \cdot \sinh \mu_0} \\ &\quad \times \sqrt{(\cosh^2 \psi - \cosh^2 \mu_0)(\cosh^2 \mu_0 - \cosh^2 \mu)(\cosh^2 \mu_0 - \cos^2 \varphi)}, \end{aligned} \tag{3.2}$$

where $h^2 = a^2 - b^2$, $0 \le \varphi < 2\pi$, $0 \le \mu \le \mu_0$, $\mu_0 \le \psi < \infty$, $\cosh^2 \mu_0 = (a^2 - c^2)/h^2$. The ellipsoid E gives rise to three coorthogonal families of the second-order confocal surfaces: $\psi = \text{const}$ is a family of confocal ellipsoids, $\mu = \text{const}$ corresponds to the hyperboloids of one sheet and $\varphi = \text{const}$ is a family of the hyperboloids of two sheets. One can obtain in the two intermediate cases either a focal ellipse $\mathcal{E}$ in the plane $z = 0$ with the equations $\psi = \mu_0$, $\mu = \mu_0$ or a focal hyperbola Γ in the plane $y = 0$ with the equations $\mu = 0$, $\varphi = 0, \pi$. The ellipsoid E has the equation $\psi = \psi_0$, where $\cosh^2 \psi_0 = a^2/h^2$; its major ellipse L with semi-axes a, b belongs to the section $z = 0$ and satisfies the equations $\psi = \psi_0$, $\mu = \mu_0$. A middle ellipse M with semi-axes a, c belongs to the plane $y = 0$ and has either equations $\psi = \psi_0$, $\mu = 0$ or $\psi = \psi_0$, $\varphi = 0 (\operatorname{mod} \pi)$ depending on the place of the points of M with respect to the branches of the focal hyperbola Γ. A minor ellipse S with semi-axes b, c has the equations $\psi = \psi_0$, $\varphi = \frac{1}{2}\pi$ and belongs to the plane $x = 0$.

One can easily find a Gaussian curvature K of the ellipsoid E:

$$\begin{aligned} K &= K(\mu, \varphi) \\ &= \frac{\cosh^2 \psi_0 \cdot \sinh^2 \psi_0}{h^2} \cdot \frac{(\cosh^2 \psi_0 - \cosh^2 \mu_0)}{(\cosh^2 \psi_0 - \cosh^2 \mu)^2 (\cosh^2 \psi_0 - \cos^2 \varphi)^2}, \end{aligned} \tag{3.3}$$

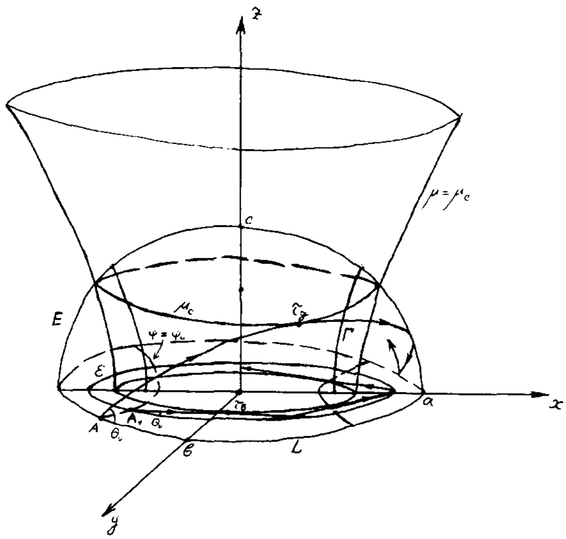

Fig. 1. The geodesic on the surface of the ellipsoid E which escapes from the point $A = \{\psi = \psi_0, \mu = \mu_0, \varphi = \varphi_0\}$ under the angle $\theta = \theta_0$ to L, and the trajectory of the billiard inside the focal ellipse $\mathcal{E}$, escaping from the point $A_1 = \{\psi = \mu_0, \mu = \mu_0, \varphi = \varphi_0\}$ under the same angle $\theta = \theta_0$ to $\mathcal{E}$, are tangent to the same common hyperboloid $\mu = \mu_c$ (the tangent points are τ_g and τ_b, respectively) (Theorem 1).

and the umbilical points of E correspond to the intersection points of E with the focal hyperbola Γ. The lines of intersection E with the hyperboloids $\mu = \text{const}$ or $\varphi = \text{const}$ are the curvature lines of the ellipsoid.

We separate the variables in the Hamilton–Jacobi equation and obtain a differential equation for the geodesic lines on the surface of ellipsoid E in the form

$$\sqrt{\frac{(\cosh^2 \psi_0 - \cosh^2 \mu)}{(\cosh^2 \mu_0 - \cosh^2 \mu)(\cosh^2 \mu - I)}}\,\mathrm{d}\mu = \sqrt{\frac{(\cosh^2 \psi_0 - \cos^2 \varphi)}{(\cosh^2 \mu_0 - \cos^2 \varphi)(I - \cos^2 \varphi)}}\,\mathrm{d}\varphi, \tag{3.4}$$

where $0 \le I \le \cosh^2 \mu_0$ is an arbitrary constant. Eq. (3.4) may be integrated in terms of the well-known two variables Theta function (Braunmühl, 1882; Moser, 1980; Staude, 1883; Veselov, 1991; Weierstrass, 1861), but one can find the results of the qualitative analysis of geodesics in Salmon 1882) (see also, Arnold (1978); Arnold et al. (1988); Cayley (1871–1873); Chasles (1846); Chenciner (1992); Darboux (1894); Douady (1982); Hilbert and Cohn-Vossen (1932); Klein (1926); Klingenberg (1982); Knörrer (1980, 1982); Roberts (1846)).

It turns out that one can consider any geodesic on the surface of E as escaping from some point $\varphi = \varphi_0$ of the oriented major ellipse L under a corresponding angle $\theta_0 \in [0, 2\pi)$ with L and this geodesic line will cross L in the next (Fig. 1) (see also, Arnold (1978);

Birkhoff (1927b); Chang and Shi (1989); Jacobi (1835); Veselov (1994)). Let (φ_1, θ_1) be the coordinates of the following intersection of this geodesic with L. Taking into account the symmetry of ellipsoid E with respect to the plane $z = 0$, we shall consider only its upper part $z \geq 0$, producing the reflection of geodesics from L according to the law: "the angle of incidence is equal to the angle of reflection". Then we glue (i.e. identify) the points of L, which are symmetric with respect to the centre of L. We thus obtain the mapping $T_g : (\varphi_0, \theta_0) \mapsto (\varphi_1, \theta_1), 0 \leq \varphi < \pi,\ 0 \leq \theta \leq \pi$, which acts on a Poincaré section $(L/\sim) \times [0, \pi]$ of geodesic flow g_t on E (see, for instance, Klingenberg (1982); Sinay (1977)); here we denote by $\sim$ the proclaimed glueing, and now the mapping T_g is in assonance with the above considered billiard mapping T in the ellipse.

A geodesic flow g_t on the ellipsoid E is an integrable one and its first integral I_t has the same form as (2.5):

$$I_t = \cosh^2 \mu \cdot \cos^2 \theta + \cos^2 \varphi \cdot \sin^2 \theta, \tag{3.5}$$

μ, φ were defined in (3.2), and θ is the intersection angle of a geodesic in question with curvature line $\mu =$ const under the corresponding value of parameter t along this geodesic (see also, Cayley (1871–1873); Joachimsthal (1843); Liouville (1844, 1846); Whittaker (1917)). Since the first integral I_t of the flow does not depend on parameter t, we obtain a first integral of the mapping T_g in the form (3.5) under $\mu = \mu_0$.

A value $I = 0$ corresponds to the elliptic fixed point of T_g or to the stable closed geodesics S of the ellipsoid E. If $0 < I < 1$, then geodesics cross the part of the ellipse M with the equations $\psi = \psi_0, \mu = 0$ and they are tangent to the curvature lines $\varphi = \pm\varphi_c \pmod{\pi}$, where $\cos^2 \varphi_c = I$. A value $I = 1$ corresponds to the geodesics going through the umbilical points of the ellipsoid (i.e. separatrices of the hyperbolic fixed point $H = (\varphi, \theta) = (0, \frac{1}{2}\pi)$ of the mapping T_g), and a point H corresponds to the unstable closed geodesics M. If $1 < I < \cosh^2 \mu_0$, then the geodesics after each reflection from L will be tangent to the curvature lines $\mu = \mu_c$, where $\cosh^2 \mu_c = I$. Finally, we obtain a stable closed geodesics L under $I = \cosh^2 \mu_0$ and the mapping T_g gives us here the transformation of the conjugate points on L.

Thus, there exists an assonance between the geodesics on the surface of the ellipsoid E and billiard trajectories inside its focal ellipse $\mathcal{E}$.

Theorem 2. *A trajectory of the billiard flow inside the focal ellipse $\mathcal{E}$ which escapes from a point $\varphi = \varphi_0$ with an angle $\theta = \theta_0$ and a geodesic line on the surface of ellipsoid E with the same initial data (i.e. escaping from a point $\varphi = \varphi_0$ of the major ellipse L under the same angle $\theta = \theta_0$ to L) are tangent either to the same common hyperboloid ($\mu =$ const or $\varphi =$ const) or both intersect the focal hyperbola Γ.*

This fact is non-obvious because there exists the spatial gap between the geodesic and billiard flows in question and a projective map $\pi : (\psi_0, \mu, \varphi) \mapsto (\mu_0, \mu, \varphi)$ of E onto its focal ellipse $\mathcal{E}$ does not conserve the angles between the curves. But Theorem 2 immediately follows from the independence of the geodesic integral (3.5) of the parameter ψ, which is a "number" of an ellipsoid from the family $\psi =$ const of the confocal ellipsoids.

As an example we present now the mapping T_g in the case $I = 1$ (see also, Braunmühl (1882); Cayley (1872); Levallois (1993); Veselov (1991)).

Lemma 3. *If $I = 1$, then the mapping T_g, which corresponds to the geodesics flow on the ellipsoid $\psi = \psi_0$, gives rise to the dynamics $\varphi_0 \mapsto \varphi_1$ on the separatrices by the formula*

$$f(\varphi_1) = \lambda_1^2 \cdot f(\varphi_0); \tag{3.6}$$

$$\lambda_1 = \exp\left[\frac{(\pi \cdot g_0 - \mathrm{i} \cdot A \cdot F(k'))}{F(k)}\right] > 1 \tag{3.7}$$

is the eigenvalue of the linear part of T_g in the hyperbolic fixed point H,

$$f(\varphi) = \frac{\Theta_1(\mathrm{i} \cdot ((g(\varphi) - g_0)/2F(k)), \tau)}{\Theta_1(\mathrm{i} \cdot ((g(\varphi) + g_0)/2F(k)), \tau)} \cdot \exp\left(\mathrm{i}A\frac{g(\varphi)}{F(k)}\right). \tag{3.8}$$

We denote

$$k' = \frac{\cosh(\mu_0)}{\cosh(\psi_0)}, \qquad k^2 = 1 - k'^2, \qquad \tau = \mathrm{i}\frac{F(k')}{F(k)},$$

$$g_0 = F\left(\arcsin\left(\frac{1}{\cosh \mu_0}\right), k'\right), \qquad g(\varphi) = F\left(\arcsin\left(\frac{\cos \varphi}{\cosh \mu_0}\right), k'\right),$$

$$A = (\partial/\partial_z)\ln \Theta_3(z, \tau)|_{z=\mathrm{i}\cdot g_0/(2F(k))},$$

formulas for $F(z, k)$, $F(k)$ and $F(k')$ were introduced in (2.7), *$\Theta_1(z, \tau)$ and $\Theta_3(z, \tau)$ are the Theta functions:*

$$\Theta_1(z, \tau) = 2 \cdot \mathrm{e}^{\pi \mathrm{i}\tau/4} \cdot \sum_{k=0}^{\infty}(-1)^k \cdot \mathrm{e}^{\pi \mathrm{i}k(k+1)\tau} \cdot \sin((2k + 1)\pi z),$$

$$\Theta_3(z, \tau) = 1 + 2 \cdot \sum_{k=1}^{\infty} \mathrm{e}^{\pi \mathrm{i}k^2\tau} \cdot \cos(2\pi kz)$$

(*Gradshteyn and Ryzhik,* 1965; *Whittaker and Watson,* 1927).

Note that $f(\varphi + \pi) = f^{-1}(\varphi)$ and A above is the pure imaginary complex quantity:

$$A = 2\pi\mathrm{i} \cdot \sum_{n=1}^{\infty}(-1)^n \frac{\sinh(\pi n g_0/F(k))}{\sinh(\pi n F(k')/F(k))},$$

the sum converges under the condition $0 \le g_0 < F(k')$. In the partial case when the minor semi-axes c of the ellipsoid tends to zero ($c \to 0$) (i.e., $\psi = \psi_0 = \mu_0$, $F(k) = \frac{1}{2}\pi$, $F(k') = \infty$), the formulas (3.6)–(3.8) turn into (2.11), i.e.

$$\tan^2\left(\tfrac{1}{2}\varphi_1\right) = \lambda_1^2 \cdot \tan^2\left(\tfrac{1}{2}\varphi_0\right), \qquad \lambda_1 = \frac{\cosh \mu_0 + 1}{\cosh \mu_0 - 1} > 1,$$

and the geodesic equation (3.4) turns into the equation of the straight line in the elliptic coordinates (2.2).

It is possible to calculate the rotation numbers for invariant curve I = const of the mapping T_g (Klingenberg, 1982; Viesel, 1971): if the rotation number is commensurable with π, then such an invariant curve corresponds to the closed geodesics on E and there exists a version of the Poncelet theorem for the corresponding geodesics (Chang and Shi, 1989).

Of course, there are no reasons for the coincidence of rotation numbers for a geodesic on the ellipsoid and for a billiard trajectory inside its focal ellipse, having the same first integral. One can easily see from (3.4) that for the orbital or trajectorial equivalence of the geodesic flows of the different ellipsoids, it is necessary to provide the similarity of these ellipsoids only (i.e. proportionality of their semi-axes a, b, c). It turns out that this condition is also a sufficient one (Bolsinov and Fomenko, 1994).

4. Ellipsoidal confocal coordinates in $\mathbb{R}^n$, $n > 3$

Let us introduce new special ellipsoidal coordinates in $\mathbb{R}^n$, $n > 3$ as being the generalization of coordinates (2.2) and (3.2). We consider $(n-1)$-dimensional ellipsoid E_{n-1} in $\mathbb{R}^n$ with the equation

$$\sum_{k=1}^{n} \frac{x_k^2}{a_k^2} = 1, \qquad 0 < a_1^2 < a_2^2 < \cdots < a_n^2, \tag{4.1}$$

and we relate the ellipsoidal coordinates $\{\mu_k\}_1^n$ with the rectangular ones $\{x_k\}_1^n$ by the formulas

$$x_{n-k+1}^2 = h^2 \cdot \frac{\prod_{m=1}^{n}(\cosh^2 \mu_m - \cosh^2 \mu_{k-1,0})}{\prod_{m=0,m\neq k-1}^{n-1}(\cosh^2 \mu_{m,0} - \cosh^2 \mu_{k-1,0})}, \tag{4.2}$$

where $k = 1, 2, \ldots, n$, $h^2 = a_n^2 - a_{n-1}^2$, $\mu_1 = \mathrm{i}\varphi$, $0 \le \varphi < 2\pi$, $\mu_{k-1,0} \le \mu_k \le \mu_{k,0}$ for $k = 2, 3, \ldots, (n-1)$, $\mu_{n-1,0} \le \mu_n < \infty$. We put in (3.2) by definition $\cosh^2 \mu_{0,0} = 0$, $\mu_{1,0} = 0$ and $\cosh^2 \mu_{k,0} = (a_n^2 - a_{n-k}^2)/h^2$ for $k = 2, 3, \ldots, (n-1)$.

The ellipsoid E_{n-1} gives rise to n coorthogonal confocal quadrics: μ_k = const are the families of the confocal $(n-1)$-dimensional hyperboloids $(1 \le k \le (n-1))$, and μ_n = const is the family of the confocal $(n-1)$-dimensional ellipsoids. The ellipsoid E_{n-1} has the equation $\mu_n = \mu_{n,0}$, where $\cosh^2 \mu_{n,0} = a_n^2/h^2$. There exist $(n-1)$-intermediate cases: $\varphi = 0 \,(\mathrm{mod}\, \pi)$, $\mu_2 = 0$ and $\mu_k = \mu_{k,0}$, $\mu_{k+1} = \mu_{k,0}$ for $k = 2, 3, \ldots, (n-1)$, which clearly define $(n-2)$-dimensional focal manifolds.

If we introduce for a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ the Jacobi elliptical coordinates $\{\lambda_k\}_1^n$ as the solutions of the equation

$$\sum_{k=1}^{n} \frac{x_k^2}{(a_k^2 - \lambda)} = 1, \tag{4.3}$$

with $\lambda_1 < a_1^2 < \lambda_2 < \ldots < \lambda_{n-1} < a_{n-1}^2 < \lambda_n < a_n^2$, then the relation of coordinates $\{\lambda_k\}_1^n$ and $\{\mu_k\}_1^n$ takes the form

$$\lambda_k = a_n^2 - h^2 \cdot \cosh^2 \mu_{n-k+1}, \qquad k = 1, 2, \ldots, n.$$

We also present a formula for the arclength ds in the ellipsoidal confocal coordinates $\{\mu_k\}_1^n$:

$$ds^2 = \sum_{k=1}^{n} g_k \cdot d\mu_k^2,$$

$$g_k = h^2 \cdot \prod_{m=1, m\neq k}^{n} (\cosh^2 \mu_k - \cosh^2 \mu_m) \Big/ \prod_{m=3}^{n} (\cosh^2 \mu_k - \cosh^2 \mu_{m-1,0}).$$

5. Conclusion

We introduced in this paper a new special representation of ellipsoidal confocal coordinates in $\mathbb{R}^n$ ($n \geq 3$) and applied them to the geodesic problem on the triaxial ellipsoid in $\mathbb{R}^3$ and to the billiard problem in its focal ellipse. We established the existence of the kind of assonance between these dynamical systems and presented some new explicit formulas for the systems in question.

Using the well-known integrals of Uhlenbeck for the surface geodesic flow and for the billiard mapping inside the general n-axis ellipsoid in $\mathbb{R}^n$ ($n \geq 3$) (Devaney, 1978; Knörrer, 1980, 1982; Moser, 1980; Veselov, 1991), taking into account (4.2), we make:

Allegation 4. *A geodesic line on the surface of general n-axis ellipsoid* $\mu_n = \mu_{n,0}$ *in* $\mathbb{R}^n$ $(n > 3)$ *and a trajectory of billiard flow inside its "focal"* $(n-1)$*-dimensional ellipsoid with the equation* $\mu_n = \mu_{n-1,0}$ *are tangent "under the same initial data" to the same confocal hyperboloids (quantity of these is equal* $(n-2)$*) from the families* $\mu_k = \text{const}$ $(1 \leq k \leq n-1)$.

It is necessary to note that from the geometric point of view our special ellipsoidal coordinates (3.2) and (4.2) are equivalent to the Jacobi ellipsoidal confocal coordinates (4.3), but (3.2) and (4.2) are sometimes more convenient than (4.3) for calculations and investigations of problems of celestial mechanics and mathematical physics (see, Arnold et al. (1988); Bykov (1965); Dubrovin et al. (1988); Knörrer (1980, 1982); Morse and Feshbach (1953); Moser (1980); Veselov (1991); Wainshtein (1965)).

Hintingly, the coordinates (3.2) (ψ, μ, φ) are linked via one-to-one correspondence with the Cartesian coordinates (x, y, z) in the whole upper $(z > 0)$ (or down $(z < 0)$) half-space (excepting only some degenerated surfaces of the smaller dimensions), whereas the Jacobi ellipsoidal coordinates $(\lambda_1, \lambda_2, \lambda_3)$ are linked via one-to-one correspondence with (x, y, z) only within octant (say, under $(x > 0,\ y > 0,\ z > 0)$).

Thus, for example, the coordinates (ψ, μ, φ) are better than (4.3) adapted to the investigation of the wave fields and eigenfunctions of Laplace operator inside the triaxial ellipsoid.

Note that Lemma 1 was used essentially in papers (Levallois and Tabanov, 1993; Tabanov, 1992, 1994) proving the non-integrability of Birkhoff's billiard in the symmetric strictly convex planar domains bounded by the perturbed *analytic* curves closed to an ellipse and for the derivation of the asymptotic formula for the transversal splitting of separatrices of the hyperbolic fixed point of the corresponding billiard mapping. In fact, the result of these papers denotes the presence of positive topological entropy for the billiard in the mentioned domains (see, Donnay (1991) for the local C^∞-smooth perturbations of the ellipse). The analogical problem on non-integrability of geodesic flows on the analytic strictly convex symmetric compact surfaces, closed to the triaxial ellipsoid in $\mathbb{R}^3$, is much more difficult one (see, Berger (1990b); Knieper and Weiss (1991); Levallois (1993); Poincaré (1892)), and our special ellipsoidal coordinates (3.2) will be probably effective when resolving this problem and for the calculation of separatrices splitting the angle.

Remark 5. Lazutkin used the KAM theory to show that for the billiard trajectories with the small angles to the boundary there exist in the phase space of strictly convex planar billiard a set of invariant curves of positive measure (Lazutkin, 1973). On the other hand, it is a very difficult problem to prove that the Lebesgue measure of the complement of the union of all KAM curves are positive and that the billiards in the strictly convex planar domains bounded by the "typical" analytic curves (or geodesic flows on the "typical" analytic strictly convex compact surfaces) have the positive metric entropy or positive Lyapunov exponents (Lazutkin, 1993). Today, it is one of the main unresolved problems of Hamiltonian dynamics with two degrees of freedom.

In the rest of this paper we pointed out that besides the ellipsoidal coordinate system there are (Morse and Feshbach, 1953) just 10 other well-known coordinate systems in $\mathbb{R}^3$, which admit the separation of variables for the reduced wave equation and for the eiconal equation of the geometrical optics; it is not difficult to see that all these coordinate systems are the special or degenerate forms of (3.2) and they may be obtained from (3.2) via different limiting transformations.

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